TECHNICAL NOTE

D-104

MINIMUM-WEIGHT ANALYSIS OF SYMMETRICAL-MULTIWEB-BEAM STRUCTURES SUBJECTED TO THERMAL STRESS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON October 1959

NATIONAL AERONAUTICS AND SPACE ADMINISTI

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SUMMARY

A minimum-weight analysis based upon buckling and yielding stresses is presented for multiweb beams subjected to thermal stress. Curves of minimum structural weight and optimum values of the beam parameters are shown as a function of bending moment for various temperature differences between skin and web. Two examples are given of the use of the curves to obtain minimum-weight beams.

INTRODUCTION

One type of construction which has been utilized in supersonic airplanes is multiweb-beam construction. The efficient design of multiweb beams has been studied by many investigators. (See, e.g., refs. 1 to 5.) Important considerations in the structural design of a modern high-speed aircraft wing are the thermal stresses caused by aerodynamic heating. Efficient beams designed to carry the combined bending and thermal stress without buckling may have structural proportions unlike the proportions for beams designed to carry only bending stress inasmuch as thermal stress increases the compressive stress in the skin, thereby reducing the allowable skin buckling load, and increases the tensile stress in the web, thereby stabilizing the web.

Inasmuch as buckling of multiweb wings may destroy their aerodynamic shape, and yielding is to be considered a structural failure, it may be desirable to avoid buckling and yielding in multiweb-wing designs. The present paper gives a solution for the minimum weight required to prevent buckling and yielding of the skin and/or web with an assumed parabolic temperature distribution across the web. Results obtained from the generalized analysis indicate the combinations of dimensions, static loads, and temperature difference between skin and web which give minimum-weight beams. In addition, relative magnitudes of the beam proportions and the trends in beam configuration for various values of heating and of bending moment may be determined from the analysis. Two examples of minimum-weight beams obtained from the analysis are given.

SYMBOLS

D	plate stiffness in bending, $\dfrac{Et_w^3}{12(1 - \mu^2)}$, in-lb
E	modulus of elasticity, psi
$K = \dfrac{4\pi^2 E}{12(1 - \mu^2)}$, psi	
M_1	bending moment per unit length of chord, in-lb/in.
T	temperature, ^{o}F
$\bar{T}$	average temperature, ^{o}F
U	external work of applied stress
V	internal energy of deformation
W	structural weight of cell per unit length of beam, lb/in.
a_n, a_m	Fourier coefficients
b	width of element, in. (fig. 1)
m	structural index, M_1/b_w^2, psi
t	thickness of element, in. (fig. 1)
w	solidity, $W/\rho b_s b_w$
x,y	beam coordinates defined in figure 1
α	coefficient of linear thermal expansion, $^{o}F^{-1}$
$\beta = (b_w/b_s)^2$	
δ	web deflection normal to web
$\gamma = t_w/b_s w$	
Λ	temperature parameter, $E\alpha(T_s - \bar{T}_w)$, psi

λ	half wave length of buckles in x-direction
μ	Poisson's ratio
σ	direct stress (positive in tension), psi
$\bar{\sigma}$	average stress over section, psi
ρ	density of beam material, lb/cu in.

Subscripts:

cr	critical
m	moment
max	maximum
min	minimum
s	compression skin
t	thermal
w	web
y	yield
1	conditions at web center line

GENERAL ANALYSIS

Multiweb beams are used as structural members in the thin wings of high-speed airplanes. It is desirable to keep the weight of these wings at a minimum and to prevent them from either buckling or yielding. In the sections to follow thermal- and bending-stress equations for a multiweb beam are developed, the critical stress in the skin and web is found, and a minimum-weight analysis is performed in which the optimum-beam proportions are found.

Assumptions

The idealized section of the symmetrical multiweb structure to be analyzed is shown in figure 1. The skin thicknesses of this beam are assumed to be small with respect to the depth so that the skin stresses and temperatures can be considered constant through the thickness. The beam depth b_w is assumed to be the distance between the skins, and the moment of inertia of the cross section is calculated by neglecting the distance from the edge of the web to the center of the skin. The beam is assumed to be of sufficient length so that all end effects may be neglected.

The idealized beam is assumed to be loaded by a static moment while the skins are receiving uniform and symmetrical heat inputs. At any instant of time the heat input is assumed to give, in addition to a uniform skin-temperature distribution, a parabolic temperature distribution across the web. Inasmuch as no joint exists between the skin and web of the idealized beam, the temperature at the edge of the web is the same as the skin temperature.

The total stress at any point in the beam is the superposed bending stress and thermal stress. The maximum total or combined stress is never allowed to exceed the elastic limit stress.

The support offered to the skins by the web is dependent upon the relative thickness of the web and skins. It is assumed in this analysis that the skins and webs are simply supported. From the results of the analysis presented in reference 1, a value of the skin critical-stress coefficient of approximately 4 may be obtained for beams of maximum efficiency. Thus, the solution for buckling under the condition of no thermal stress obtained from the present analysis is in agreement with the similar solution obtained in reference 1.

No crushing or shearing forces are assumed to be acting on the webs and the webs are assumed to buckle in waves, the nodal lines of which are parallel to the y-axis. Furthermore, the web is assumed to buckle under the action of the bending stress and the average web thermal stress inasmuch as the analysis based upon the average web thermal stress gives a conservative value for the critical stress when compared with the value of critical stress obtained using a parabolic web-thermal-stress distribution.

The material properties of the beam are assumed invariant with temperature. In addition, the material is assumed perfectly elastic up to the yield point.

Fundamental Equations

Thermal-stress equations.- The thermal stress at any location on the cross section, based upon elementary beam assumptions, is given by

$$\sigma_t = -E\alpha(T - \bar{T}) \tag{1}$$

where T is the temperature at a particular location on the cross section, and $\bar{T}$ is the average temperature of the cross section. The value of the skin thermal stress may then be expressed in terms of the skin temperature T_s, average web temperature $\bar{T}_w$, and beam dimensions by expressing $\bar{T}$ in terms of these parameters as

$$\bar{T} = \frac{2t_s b_s T_s + t_w b_w \bar{T}_w}{2t_s b_s + t_w b_w} \tag{2}$$

The skin thermal stress and average web thermal stress are given, respectively, by

$$\sigma_{s,t} = -\frac{\Lambda}{1 + \frac{2t_s b_s}{t_w b_w}} \tag{3}$$

and

$$\bar{\sigma}_{w,t} = \frac{2\Lambda \frac{t_s b_s}{t_w b_w}}{1 + \frac{2t_s b_s}{t_w b_w}} \tag{4}$$

where

$$\Lambda = E\alpha\left(T_s - \bar{T}_w\right) \tag{5}$$

The parabolic temperature distribution across the web may be expressed as

$$T_w = \left(T_s - T_1\right)\left(\frac{2y}{b_w} - 1\right)^2 + T_1 \tag{6}$$

where T_s is the skin temperature and T_1 the temperature at the web center line. Similarly, the web thermal stress is given by

$$\sigma_{w,t} = \left(\sigma_{s,t} - \sigma_1\right)\left(\frac{2y}{b_w} - 1\right)^2 + \sigma_1 \tag{7}$$

where σ_1 is the thermal stress at the web center line. The average value of the web thermal stress found from equation (7) is expressed by

$$\bar{\sigma}_{w,t} = \frac{1}{3}\left(\sigma_{s,t} + 2\sigma_1\right) \tag{8}$$

The expression for σ_1 is found in terms of the beam proportions by substituting the right-hand sides of equations (3) and (4) into equation (8). This expression is given by

$$\sigma_1 = \frac{\Lambda\left(1 + \frac{6t_s b_s}{t_w b_w}\right)}{2\left(1 + \frac{2t_s b_s}{t_w b_w}\right)} \tag{9}$$

Bending-stress equations.- The stresses in the cross section due to an external bending moment are found by using elementary beam theory and are given by

$$\sigma_m = \frac{m\left(1 - \frac{2y}{b_w}\right)}{\frac{t_s}{b_w} + \frac{1}{6}\frac{t_w}{b_s}} \tag{10}$$

where m is the structural index defined as

$$m = \frac{M_i}{{b_w}^2} \tag{11}$$

where M_i is the external moment per unit length of chord. The stress in the compression skin due to an external moment is then given by

$$\sigma_{s,m} = -\frac{m}{\frac{t_s}{b_w} + \frac{1}{6}\frac{t_w}{b_s}} \tag{12}$$

Critical-stress equations.- Inasmuch as the skin is assumed to be simply supported, the skin buckling stress is given by

$$\sigma_{cr} = -\frac{4\pi^2 E}{12\left(1 - \mu^2\right)}\left(\frac{t_s}{b_s}\right)^2 \tag{13}$$

The bending stress required to buckle the skin is found by equating the skin buckling stress to the combined thermal and bending compressive stresses in the skin and solving for the compressive skin bending stress $\sigma_{s,m}$. This expression for $\sigma_{s,m}$ is given by

$$\sigma_{s,m} = -\frac{4\pi^2 E}{12\left(1 - \mu^2\right)}\left(\frac{t_s}{b_s}\right)^2 + \frac{\Lambda}{1 + \frac{2t_s b_s}{t_w b_w}} \tag{14}$$

The equation governing web buckling under the combined stresses is derived in the appendix. From this analysis the following approximate expression for the critical bending stress in the web is obtained

$$\sigma_{s,m} = -24.98 \frac{\pi^2 E}{12\left(1 - \mu^2\right)} \left(\frac{t_w}{b_w}\right)^2 - 5.55 \frac{\Lambda \dfrac{t_s b_s}{t_w b_w}}{1 + \dfrac{2t_s b_s}{t_w b_w}} \tag{15}$$

Note that $\sigma_{s,m}$ is also the bending stress at the edge of the web.

MINIMUM-WEIGHT ANALYSES

The expression "minimum-weight beam" commonly denotes the beam of the lightest weight required to support a given load. However, if a given weight and beam depth are assumed, it also may be thought of as the beam configuration that will support the largest moment for that weight. In the present study the latter concept is used and the maximum applied moment is found with respect to the remaining beam dimensions. Two analyses are presented and are outlined as follows.

In the first analysis, the usual procedure, which allows the skin and web to buckle simultaneously, is used to obtain a minimum-weight solution. However, beams obtained from this solution are not capable of supporting the applied moment in the absence of thermal stress (uniform temperature conditions) because thermal stresses make it possible for the web thickness to be reduced to values below the value required to support the applied moment under uniform temperature conditions.

Inasmuch as the minimum-weight beams should be capable of supporting the applied moment in the absence of thermal stress, the second analysis requires that the webs be of sufficient thickness so that the beam will support the applied moment under uniform temperature conditions and under all intermediate values of thermal stress below the design value. In addition, this analysis includes a yielding criterion for the skin and web.

First Analysis

The weight of the structure per unit length is proportional to the area of the cross section inasmuch as the beam is assumed to be constructed of one material. The weight equation may then be written as

$$W = \rho\left(2t_s b_s + t_w b_w\right) \tag{16}$$

The weight may be expressed for convenience in terms of the solidity of the cross section by

$$w = \frac{W}{\rho b_s b_w} \tag{17}$$

The expression for solidity then becomes

$$w = 2\,\frac{t_s}{b_w} + \frac{t_w}{b_s} \tag{18}$$

For the purpose of simplifying the procedure, $\sigma_{s,m}$ in equation (14) can be expressed, with the use of equations (12) and (18), as

$$\frac{m}{K} = w\left(\frac{1}{2} - \frac{\gamma}{3}\right)\left[\frac{w^2}{4}\,(1 - \gamma)^2\beta - \frac{\Lambda}{K}\,\gamma\right] \tag{19}$$

where K is defined as $\dfrac{4\pi^2 E}{12\left(1 - \mu^2\right)}$ and where γ and β are parameters defined as follows:

$$\gamma = \frac{t_w}{b_s}\,\frac{1}{w} \tag{20}$$

$$\beta = \left(\frac{b_w}{b_s}\right)^2 \tag{21}$$

It should be noted that the beam parameters γ, β, and w completely specify the beam proportions. (The skin thickness can be found through the use of equation (18).)

As the first approach to the problem of finding the minimum weight, the skin and web are assumed to buckle simultaneously for a given value of Λ/K. This assumption imposes the requirement that the critical bending stress in the web be equal to the skin critical bending stress and is satisfied by equating the critical-bending-stress equations for the skin and web (eqs. (14) and (15)). The resulting expression, written in terms of γ, β, and Λ/K is given by

$$\frac{w^2}{4}\,(1 - \gamma)^2\beta - \frac{\Lambda}{K}\,\gamma = 6.237\,\frac{w^2\gamma^2}{\beta} + 2.772\,\frac{\Lambda}{K}\,(1 - \gamma) \tag{22}$$

The maximum applied moment is found by means of Lagrangian multipliers using equation (22) as the constraining relation and equation (19) as the moment equation. Thus, the constrained-maximum moment that is applied is found by varying the dimensionless structural index m/K with respect to γ and β while keeping w constant. This operation gives the following equation which must be solved with equation (22) to obtain the beam parameters:

$$\left[\beta^2(1-\gamma)(14.63\gamma^2-37.81\gamma+25.17)+\gamma^2(99.79\gamma-74.84)\right]\left[\beta^2(1-\gamma)^2+24.95\gamma^2\right]$$
$$=\beta^2(1-\gamma)^2(3-2\gamma)\left[\beta^2(1-\gamma)(3.77-1.77\gamma)-\gamma(44.21\gamma-138.31)\right] \qquad (23)$$

Inasmuch as equation (23) is difficult to solve explicitly for substitution into equation (22), given values of γ are chosen and β is then determined from equation (23). The corresponding value of the solidity w is then found from equation (22) for a given value of Λ/K, and, finally, the corresponding value of m/K is found from equation (19).

Curves of m/K plotted against w for various values of Λ/K are shown in figure 2. From this figure it is apparent that for a given structural index and temperature differential the most efficient beam is lighter than the most efficient beam with no temperature differential. This rather surprising reduction in structural weight is the result of the stabilizing effect upon buckling of the thermal-tensile stress in the web which makes it possible to decrease the web thickness to very low values. Under uniform temperature conditions these beams would not take the applied load because of web buckling.

The dashed line in figure 2 indicates the limit of all constrained extremums obtained from the analysis. Below this dashed line no constrained extremums exist. However, equations (19) and (22) show that below the dashed line the weight decreases monotonically with decreasing web thickness. Thus, in the limit, the lightest weight beam would have zero web thickness and the assumption of the analysis would obviously not apply for such beams. Therefore, the curves of figure 2 are terminated at the dashed line.

Inasmuch as only two terms of the web-deflection equation are used in determining the web-buckling equation, the curves of figure 2 are good only for indicating trends of the weight and moment for small values of Λ/K. In addition, in this analysis only elastic conditions are considered; yielding of the beam has not been considered.

Second Analysis

The beams obtained from the second solution differ from those obtained in the previous solution inasmuch as they will support any value of thermal stress below the design value. Yielding of the structure is also considered in this analysis. From the carpet plot of figure 3 it may be seen that the resulting solution is separated into four regions. Each region represents an area in which the stress level in the skin or web is either below or at the yield stress. In region 1 stresses in the skin and web are below the yield stress. As the moment is increased, the combined compressive stress in the skin reaches the yield stress. Curves for which the skin stress is equal to the yield stress are shown in region 2. As the thermal stress is increased, the combined tensile stress in the web reaches the yield stress. Curves for which the web stress is equal to the yield stress are shown in region 3. With increasing moment and thermal stress both the compressive stress in the skin and tensile stress in the web eventually reach the yield stress. Curves for which both the skin and web stresses are equal to the yield stress are shown in region 4.

Buckling below yield stress.- The webs must have sufficient thickness to withstand the applied loads under uniform temperature conditions. Therefore, in this analysis, the effects of thermal stress on web buckling are excluded. The effects of thermal stress upon skin buckling are included as before. Equating allowable bending stresses for skin buckling (eq. (14)) and web buckling (eq. (15) with $\Lambda = 0$) yields

$$w^2\left[\frac{\beta^2}{4}(1 - \gamma)^2 - 6.237\gamma^2\right] - \gamma\beta\frac{\Lambda}{K} = 0 \tag{24}$$

The maximum applied moment is found by the same procedure used in the previous analysis, that is, by using equation (24) as the constraining relation instead of equation (22) and by using equation (19) as the moment equation. This procedure gives the following equation which must be solved with equation (24) to obtain the beam parameters:

$$\beta = 6\gamma\sqrt{\frac{0.693(3 - 4\gamma)}{(\gamma - 1)(8\gamma^2 - 13\gamma + 3)}} \tag{25}$$

Values of γ are chosen and the corresponding values of β, w, and m/K are found from equations (25), (24), and (19), respectively, for constant values of Λ/K. The corresponding curves of m/K plotted against w for values of Λ/K are shown in region 1 in figure 3.

Skin buckling stress at yield.- In region 1 the proportions of the optimum beams vary so that the skin buckling stress increases with increasing applied moment and will eventually reach the compressive yield stress. After the skin buckling stress reaches the compressive yield stress the expression for the skin buckling stress becomes, from equation (13),

$$\frac{\sigma_y}{K} = \left(\frac{t_s}{b_s}\right)^2 \tag{26}$$

The solidity equation (eq. (18)) may then be used to eliminate the skin-thickness parameter from equation (26) which gives

$$w^2 = \frac{4\frac{\sigma_y}{K}}{\beta(1 - \gamma)^2} \tag{27}$$

Equation (27) can be substituted into equation (24) to give β as a function of γ and Λ/K as follows:

$$\beta = \frac{5\gamma}{1 - \gamma} \frac{1}{\sqrt{1 - \gamma\frac{\Lambda/K}{\sigma_y/K}}} \tag{28}$$

Values of γ may be chosen and the corresponding values of β, w, and m/K are found from equations (28), (27), and (19), respectively. The corresponding curves of m/K plotted against w for values of Λ/K are shown in region 2 in figure 3. The value of σ_y/K is assumed to be the same for both tension and compression and is taken to be 0.0016. This value was not intended to be characteristic of a given material but is at the high end of the range of σ_y/K values for materials such as titanium, Inconel X, and aluminum.

Web tensile stress at yield.- As the temperature differential between skins and web increases in region 1, the web is subjected to larger tensile stress and the maximum web stress eventually reaches the yield stress. The web stress at any point may be written as the summation of the parabolic thermal stress distribution and the bending stress distribution and is given by

$$\sigma_w = \left(\sigma_{s,t} - \sigma_1\right)\left(\frac{2y}{b_w} - 1\right)^2 + \sigma_1 + \sigma_{s,m}\left(\frac{2y}{b_w} - 1\right) \tag{29}$$

The location of the maximum web stress is found by setting the derivative of σ_w in equation (29) with respect to y equal to zero. Substituting the resulting value of y into equation (29) gives the maximum web stress as

$$\sigma_{w,max} = \sigma_1 - \frac{\sigma_{s,m}^2}{4(\sigma_{s,t} - \sigma_1)} \tag{30}$$

for

$$\frac{\sigma_{s,m}}{2\left(\sigma_{s,t} - \sigma_1\right)} < 1 \tag{31}$$

This condition is necessary in order that the mathematical maximum occur on the web.

Expressing equation (30) in terms of the beam parameters using equations (3), (9), and (14) and setting $\sigma_{w,max}$ equal to the yield stress gives

$$\left[\frac{w^2(1 - \gamma)^2\beta}{4} - \frac{\Lambda}{K}\gamma\right]^2 + \left(\frac{\Lambda}{K}\right)^2(9 - 6\gamma) = 6\frac{\sigma_y}{K}\frac{\Lambda}{K} \tag{32}$$

Equations (24) and (32) may be combined to give

$$\beta^2 = \frac{14.4\gamma^2\left(\gamma + 1.732\sqrt{2\gamma - 3 + \frac{2\frac{\sigma_y}{K}}{\Lambda/K}}\right)}{(1 - \gamma)^2\sqrt{2\gamma - 3 + \frac{2\frac{\sigma_y}{K}}{\Lambda/K}}} \tag{33}$$

Values of γ are chosen and the corresponding values of β, w, and m/K are found from equations (33), (32), and (19), respectively. The corresponding curves of m/K plotted against w for values of Λ/K are shown in region 3 in figure 3.

Skin buckling stress and web tensile stress at yield.- The maximum skin stress in region 3 and the maximum web stress in region 2 will eventually reach the yield point with increasing applied moment and increasing thermal stress. The expression for w that was obtained for skin stress at yield (eq. (27)) may be solved for β and substituted into the expression for the web stress at yield (eq. (32)). The beam parameter γ may then be found from equation (32) as a function of Λ/K and is given by

$$\gamma = 3 + \frac{\sigma_y/K}{\Lambda/K} - 2\sqrt{\frac{3\frac{\sigma_y}{K}}{\Lambda/K}} \tag{34}$$

The expression for γ from equation (34) and the expression for β from equation (27) may be substituted into equation (19) to obtain the following equation for m/K as a function of w and Λ/K:

$$\frac{m}{K} = w\left[3\,\frac{\Lambda}{K} - 0.13856\left(\frac{\Lambda}{K}\right)^{1/2}\right]\left[\frac{0.000533}{\Lambda/K} - \frac{0.04619}{(\Lambda/K)^{1/2}} + \frac{1}{2}\right] \tag{35}$$

These curves are shown in region 4 in figure 3.

Optimum-beam proportions.- The beam proportions for minimum-weight beams are shown in figures 4, 5, and 6 and are obtained directly from the beam parameters γ, β, and w. In regions 1, 2, and 3, the beam parameters β and w are found for given values of γ and Λ/K as previously discussed. The web spacing parameter b_w/b_s may then be found from equation (21). The web thickness parameter t_w/b_w is given by

$$\frac{t_w}{b_w} = \frac{\gamma w}{b_w/b_s} \tag{36}$$

The skin-thickness proportion t_s/b_w is found by using equation (18). Note that the proportion t_w/b_s is simply the product of γ and w.

The optimum-beam proportions for region 4 are found for given values of w and Λ/K. Note that the value of γ is constant for a given value of Λ/K. (See eq. (34).) The value of β is then determined by equation (27). The optimum values of the beam proportions b_w/b_s, t_w/b_w, and t_s/b_w are then found by the method outlined in the preceding paragraph.

Examples of beams obtained using curves.- Beam proportions for minimum weight may be found by using the curves of figures 4 to 6. The values of m/K and b_w are assumed to have been determined from aerodynamic considerations. The value of Λ/K may be determined from a trial-and-error procedure as follows. The optimum-beam proportions for a value of Λ/K of zero are found and then the temperature distribution and actual value of Λ/K are calculated for this beam. The new value of Λ/K is then used and the procedure repeated until Λ/K converges.

As an example, figures 4 to 6 are used to determine the proportions of two Inconel X beams. One beam is designed for uniform temperature $\left(\frac{\Lambda}{K} = 0\right)$ and the other, for an average-temperature difference $\left(T_s - \bar{T}_w\right)$ of 550° F. The moment per inch of chord M_1 to be supported by both beams is given as 55,500 in-lb/in. and the wing depth b_w is 4 inches. If the material properties are taken as $E = 31 \times 10^6$ psi and $\alpha = 7.6 \times 10^{-6}$ in./in./$^{\circ}$F, the value of the temperature parameter for the beam with thermal stress is $\frac{\Lambda}{K} = 0.001156$ and the value of the dimensionless structural index for both beams is $\frac{m}{K} = 0.031 \times 10^{-3}$. The solidity of each beam is found from figure 3 (see circular symbols) and is given by $w = 0.065$ for $\frac{\Lambda}{K} = 0$ and $w = 0.069$ for $\frac{\Lambda}{K} = 0.001156$. The weight of each beam is then calculated as 11.22 psf for $\frac{\Lambda}{K} = 0$ and 11.92 psf for $\frac{\Lambda}{K} = 0.001156$. Values of the beam proportions for the two values of Λ/K are found from the curves of figures 4 to 6 and are as follows:

For $\frac{\Lambda}{K} = 0$,

$$\frac{b_w}{b_s} = 1.79$$

$$\frac{t_w}{b_w} = 0.0144$$

$$\frac{t_s}{b_w} = 0.02$$

and for $\frac{\Lambda}{K} = 0.001156$,

$$\frac{b_w}{b_s} = 1.82$$

$$\frac{t_w}{b_w} = 0.0138$$

$$\frac{t_s}{b_w} = 0.0218$$

The corresponding values of the beam dimensions are as follows:

For $\frac{\Lambda}{K} = 0$,

$$b_s = 2.235$$

$$t_w = 0.0576$$

$$t_s = 0.08$$

and for $\frac{\Lambda}{K} = 0.001156$,

$$b_s = 2.199$$

$$t_w = 0.0552$$

$$t_s = 0.0872$$

Note that the beam designed for thermal stress is approximately $6\frac{1}{4}$ percent heavier than the uniform-temperature design.

Web Crushing

The forces tending to crush the webs have not been considered in the preceding analyses. The crushing stress may be found by equating the y-component of the force in the skin due to bending to the Euler load required to buckle a simply supported plate. This equality is given by

$$\frac{2\sigma_{s,m}{}^2 t_s b_s}{E b_w t_w} = 0.9E\left(\frac{t_w}{b_w}\right)^2 \tag{37}$$

The magnitude of the bending stress required to crush the webs as determined from equation (37), is greater than the applied bending stress for beams considered in the second analysis. Web crushing, therefore, would not affect this analysis.

In the first analysis, in which the web thickness decreases to relatively small values, web crushing would occur in beams in which thermal stress is present and the value of the moment is small.

DISCUSSION

Two solutions based upon buckling criterion for minimum-weight multiweb beams have been presented and curves giving the structural weight required to prevent either buckling or yielding of these beams are shown for values of the moment parameter and temperature parameter. The first solution has little practical application, inasmuch as a beam designed for a given value of m/K and a given value of temperature difference Λ/K will fail when subjected to a smaller temperature differential. Failure will occur for values of the temperature parameter below the given value inasmuch as web thermal stress stabilizes the web and thicker webs are required for lower values of the temperature parameter Λ/K. This solution, however, indicates that structural-weight reductions with respect to beams designed for uniform temperature and maximum bending stress may be possible in some designs where the thermal stress and bending stress are always increasing.

The second solution has broader applications and indicates that additional beam material is required to support a given applied moment when the beam is subjected to thermal stress. It was found that a beam designed for given values of m/K and Λ/K could support the combined stress for all values of Λ/K less than the design value. The weight penalty due to thermal stress may be found from figure 3 by measuring the vertical shift of the minimum-weight curves between different values of Λ/K and for a given value of m/K. The rate of weight change with temperature differences for a given moment is also indicated in figure 3. It is apparent that a large weight penalty due to thermal stress is incurred in regions 3 and 4. Curves showing the ratios of beam depth to web spacing, web thickness to beam depth, and skin thickness to beam depth are shown in figures 4, 5, and 6, respectively. In region 1 the weight penalty is caused by the increase in skin thickness required to keep the skin from buckling. The web thickness and spacing decrease in this region for a given moment and for increasing values of Λ/K. With increasing moment the skin and web thicknesses increase along with the web spacing. In region 2 the weight penalty is again caused by the increase in skin thickness required. The web thickness decreases and the web spacing increases for a given moment and increasing values of Λ/K. In region 3 the weight penalty is caused by the increase in web weight requirements inasmuch as the webs become thicker and more closely spaced while the skin thickness decreases. In region 4, where the weight penalty for thermal stress is very severe, the additional weight is caused both by the additional skin thickness required and by thicker webs, even though the webs are spaced farther apart. It has been found that the thermal stress in optimum beams does not vary directly as the temperature parameter; however, the thermal stress increases in all regions of figure 3 with increasing values of Λ/K for either a given value of w or m/K. In region 4 the magnitude of the thermal stresses does not change for a given value of Λ/K inasmuch as $t_s b_s / t_w b_w$ remains constant.

The weight penalties show the importance of using a design that keeps the thermal stress at a minimum. Inasmuch as web weight requirements incurred by the thermal stress contribute greatly to the weight penalty, web designs other than that presented herein may result in smaller weight penalties. Designs in which the web is allowed either to deform plastically or to expand freely, as may be the case if corrugated webs are used, would probably result in a less severe weight penalty.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., July 22, 1959.

APPENDIX

DETERMINATION OF WEB CRITICAL BENDING STRESS

In order to determine the amount of bending stress required to buckle the web, the effect of thermal stresses on web buckling must be considered. The solution for the buckling of a simply supported web under a combined bending and uniform stress distribution is given in reference 6 (p. 350). The following analysis presents a similar solution for the web under a combined bending and parabolic stress distribution.

It is assumed in this analysis that the web will buckle in waves the nodal lines of which are parallel to the y-axis. The deflection of the web over a half wave length is then given by

$$\delta = \sin\frac{\pi x}{\lambda_w}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b_w} \tag{A1}$$

where λ_w is the half wave length along the buckle pattern. The web stress due to the combined loading is given by

$$\sigma_w = \left(\sigma_{s,t} - \sigma_1\right)\left(\frac{2y}{b_w} - 1\right)^2 + \sigma_1 + \sigma_{s,m}\left(\frac{2y}{b_w} - 1\right) \tag{A2}$$

where $\sigma_{s,t}$ is the compressive thermal stress in the skin and $\sigma_{s,m}$ is the compression skin bending stress.

The work done by the external forces during buckling, if no crushing and shear forces are assumed, is given by (ref. 6, p. 314, eq. (201))

$$U = -\frac{t_w}{2}\iint \sigma_w\left(\frac{\partial\delta}{\partial x}\right)^2 dx\, dy \tag{A3}$$

where σ_w is assumed positive in tension. The strain energy for bending of the web during buckling over a half wave length is given by (ref. 6, p. 307, eq. (199))

$$V = \frac{D}{2} \int_0^{b_w} \int_0^{\lambda_w} \left\{ \left(\frac{\partial^2 \delta}{\partial x^2} + \frac{\partial^2 \delta}{\partial y^2} \right)^2 - 2(1 - \mu) \left[\frac{\partial^2 \delta}{\partial x^2} \frac{\partial^2 \delta}{\partial y^2} - \left(\frac{\partial^2 \delta}{\partial x \, \partial y} \right)^2 \right] \right\} dx \, dy \tag{A4}$$

After the energy expressions U and V are integrated between the limits and set equal to each other, the following expression is obtained (in this equation, m and p represent integers):

$$\begin{aligned} \frac{D\lambda_w^2 \pi^4}{8} \sum_{n=1}^{\infty} a_n^2 \left(\frac{1}{\lambda_w^2} + \frac{n^2}{b_w^2} \right)^2 = {} & 4t_w(\sigma_1 - \sigma_{s,t}) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m mn}{(n^2 - m^2)^2} \\ & - 2t_w \sigma_{s,m} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{a_n a_p pn}{(n^2 - p^2)^2} \\ & - \frac{\pi^2}{4} t_w \left(\frac{\sigma_1}{3} + \frac{\sigma_{s,t}}{6} \right) \sum_{n=1}^{\infty} a_n^2 \\ & - \frac{t_w}{4} (\sigma_1 - \sigma_{s,t}) \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} \end{aligned} \tag{A5}$$

where $n \neq m$ and $n + m$ is even in the first summation and in the second summation $n \neq p$ and $n + p$ is odd. If the first two terms of the summations in equation (A5) are used the following expression is obtained:

$$\begin{aligned} \frac{D\pi^4}{8\lambda_w^2} \left[a_1^2 \left(1 + \frac{\lambda_w^2}{b_w^2} \right)^2 + a_2^2 \left(1 + \frac{4\lambda_w^2}{b_w^2} \right)^2 \right] = {} & -\frac{8}{9} a_1 a_2 t_w \sigma_{s,m} \\ & - \frac{\pi^2 t_w}{4} \left(a_1^2 + a_2^2 \right) \left(\frac{\sigma_1}{3} + \frac{\sigma_{s,t}}{6} \right) \\ & - \frac{t_w}{4} (\sigma_1 - \sigma_{s,t}) \left(a_1^2 + \frac{a_2^2}{4} \right) \end{aligned} \tag{A6}$$

The coefficients a_1 and a_2 must be adjusted so that $\sigma_{s,m}$ will be a minimum for a given value of λ_w / b_w. This minimization may be made by setting the partial derivatives of equation (A6) with respect to

a_1 and a_2 equal to zero. Two homogeneous linear equations are obtained which yield a nontrivial solution and give a minimum value of $\sigma_{s,m}$ for a given value of λ_w/b_w. This minimum value is obtained by the following equation:

$$\sigma_{s,m}{}^2 = \frac{81\pi^4}{1024}\left[\frac{\left(1 + \frac{\lambda_w{}^2}{b_w{}^2}\right)^2}{\frac{\lambda_w{}^2}{b_w{}^2}} \frac{\pi^2 D}{b_w{}^2 t_w} + \frac{2}{\pi^2}\left(\sigma_1 - \sigma_{s,t}\right)\right.$$

$$\left. + \bar{\sigma}_{w,t}\right]\left[\frac{\left(1 + 4\frac{\lambda_w{}^2}{b_w{}^2}\right)^2}{\frac{\lambda_w{}^2}{b_w{}^2}} \frac{\pi^2 D}{b_w{}^2 t_w} + \frac{\sigma_1 - \sigma_{s,t}}{2\pi^2} + \bar{\sigma}_{w,t}\right] \tag{A7}$$

A similar expression may be obtained by using the average web thermal stress and reference 6 (p. 354, eq. (i)). For the present analyses the numerical factor α, which is presented in reference 6, is defined by the following equation:

$$\alpha = -\frac{2\sigma_{s,m}}{\bar{\sigma}_{w,t} - \sigma_{s,m}} \tag{A8}$$

Note that in the analysis presented in reference 6 compression is assumed to be positive. The resulting expression is

$$\sigma_{s,m}{}^2 = \frac{81\pi^4}{1024}\left[\frac{\left(1 + \frac{\lambda_w{}^2}{b_w{}^2}\right)^2}{\frac{\lambda_w{}^2}{b_w{}^2}} \frac{\pi^2 D}{b_w{}^2 t_w} + \bar{\sigma}_{w,t}\right]\left[\frac{\left(1 + 4\frac{\lambda_w{}^2}{b_w{}^2}\right)^2}{\frac{\lambda_w{}^2}{b_w{}^2}} \frac{\pi^2 D}{b_w{}^2 t_w} + \bar{\sigma}_{w,t}\right] \tag{A9}$$

When equations (A7) and (A9) are compared it is apparent that a value of critical web stress calculated from equation (A9) would be lower than a corresponding value calculated from equation (A7). Therefore, in order to simplify the present analysis, the conservative value of the critical web stress obtained from equation (A9) is used. In the absence of thermal stress the value of the critical-web-stress coefficient, obtained from equation (A9), is within 6 percent of the value obtained for the zero-end-restraint condition of reference 1. The critical stress may be found by setting

$$\frac{\partial \sigma_{s,m}{}^{2}}{\partial\left(\frac{\lambda_w}{b_w}\right)^{2}} = 0 \tag{A10}$$

from which

$$\bar{\sigma}_{w,t} = \frac{\pi^{2}E}{12\left(1 - \mu^{2}\right)}\left(\frac{t_w}{b_w}\right)^{2} \frac{2\left[4\left(\frac{\lambda_w}{b_w}\right)^{2} + 1\right]\left[\left(\frac{\lambda_w}{b_w}\right)^{2} + 1\right]\left[4\left(\frac{\lambda_w}{b_w}\right)^{4} - 1\right]}{\left(\frac{\lambda_w}{b_w}\right)^{2}\left[2 - 17\left(\frac{\lambda_w}{b_w}\right)^{4}\right]} \tag{A11}$$

Substituting equation (A11) into equation (A9) gives the expression for $\sigma_{s,m,min}$. Solving for $\sigma_{s,m,min}$ and $\bar{\sigma}_{w,t}$ for various values of λ_w/b_w and plotting $\sigma_{s,m,min}$ against $\bar{\sigma}_{w,t}$ gives a curve which may be closely approximated by

$$\sigma_{s,m} = -24.98 \frac{\pi^{2}E}{12\left(1 - \mu^{2}\right)}\left(\frac{t_w}{b_w}\right)^{2} - 2.775\, \bar{\sigma}_{w,t} \tag{A12}$$

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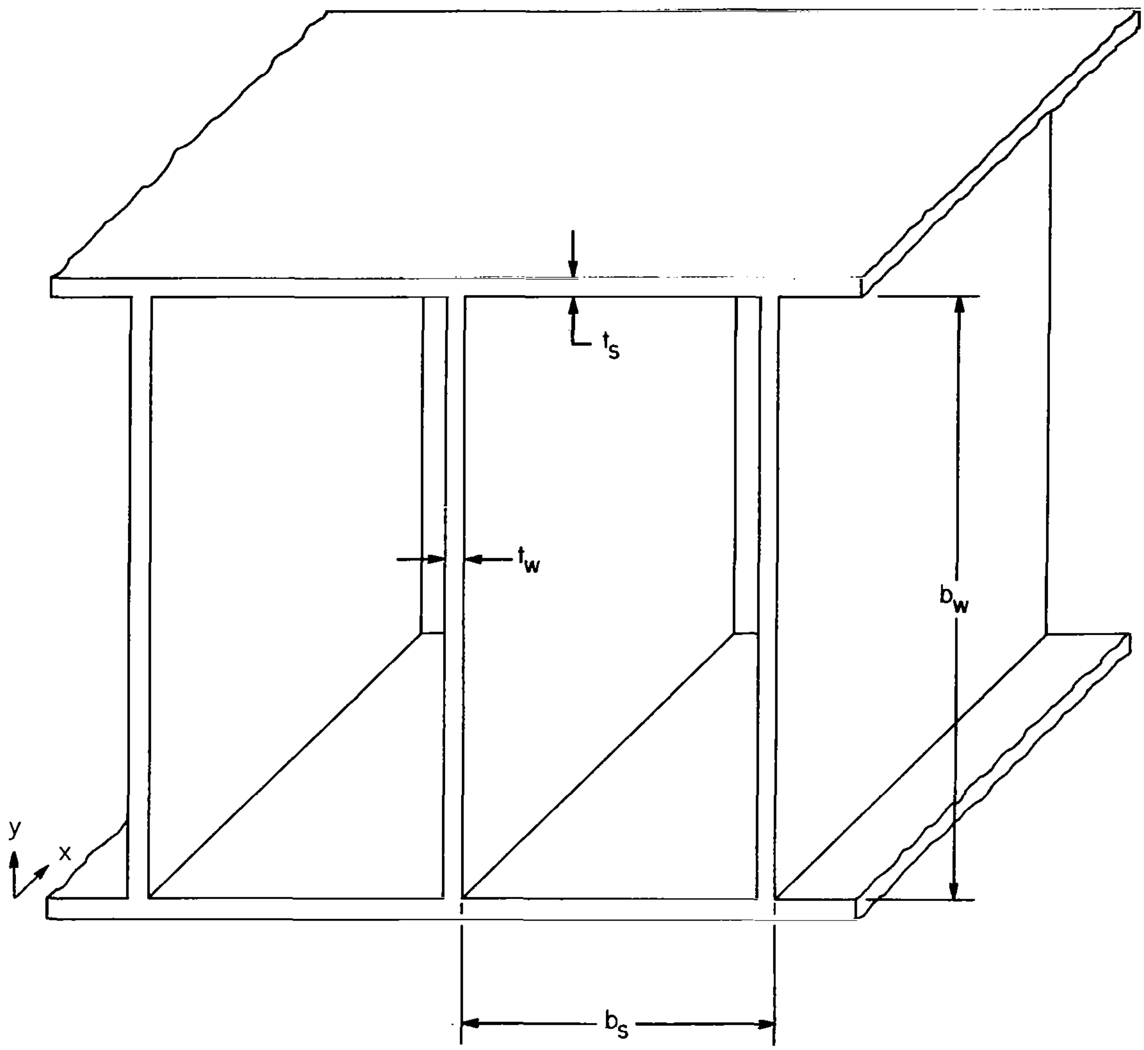


Figure 1.- Idealized section of multiweb beam.

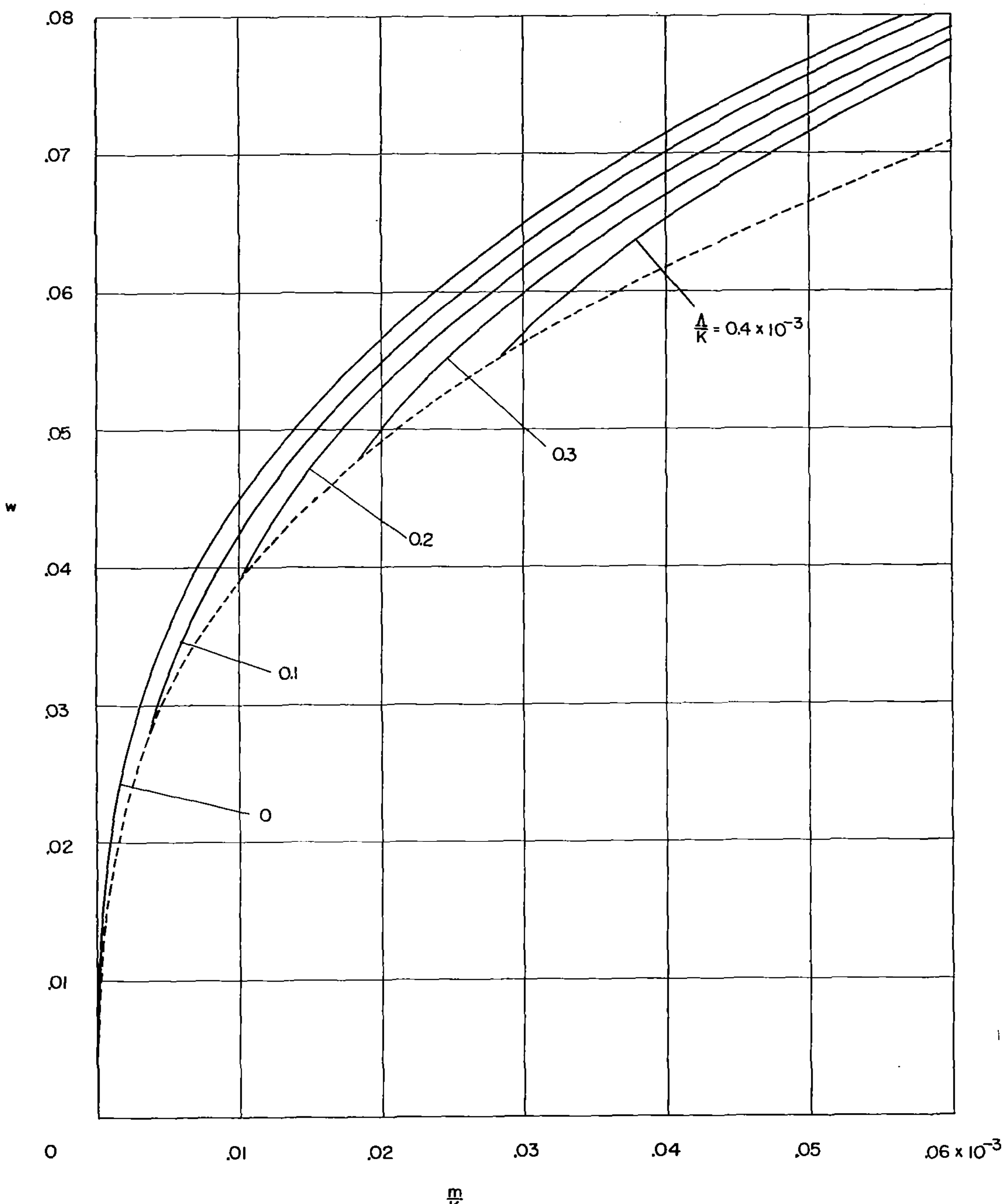


Figure 2.- Solidity as a function of dimensionless structural index for optimum beams obtained from first analysis. Dashed line indicates limit of all constrained extremums.

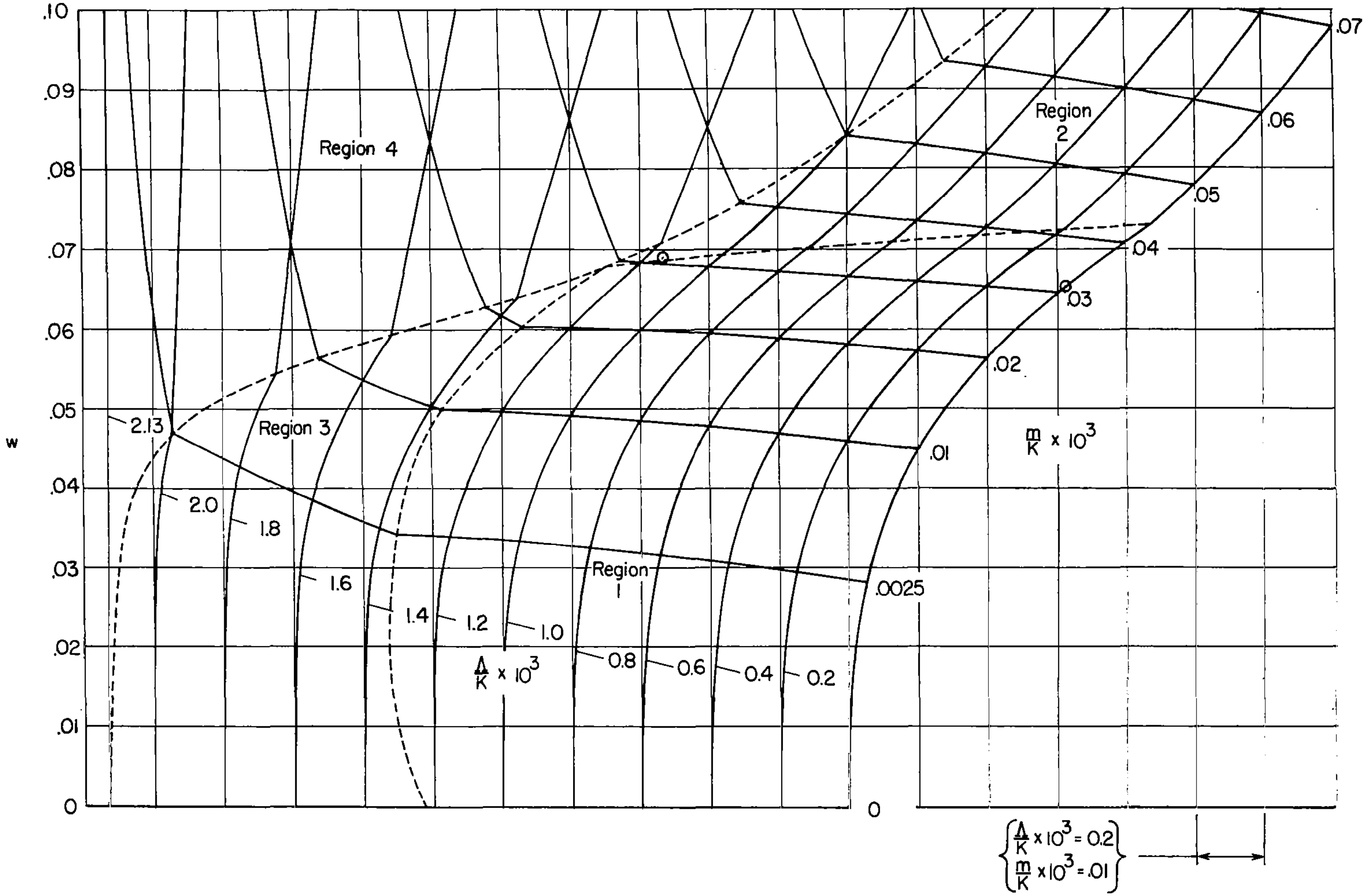
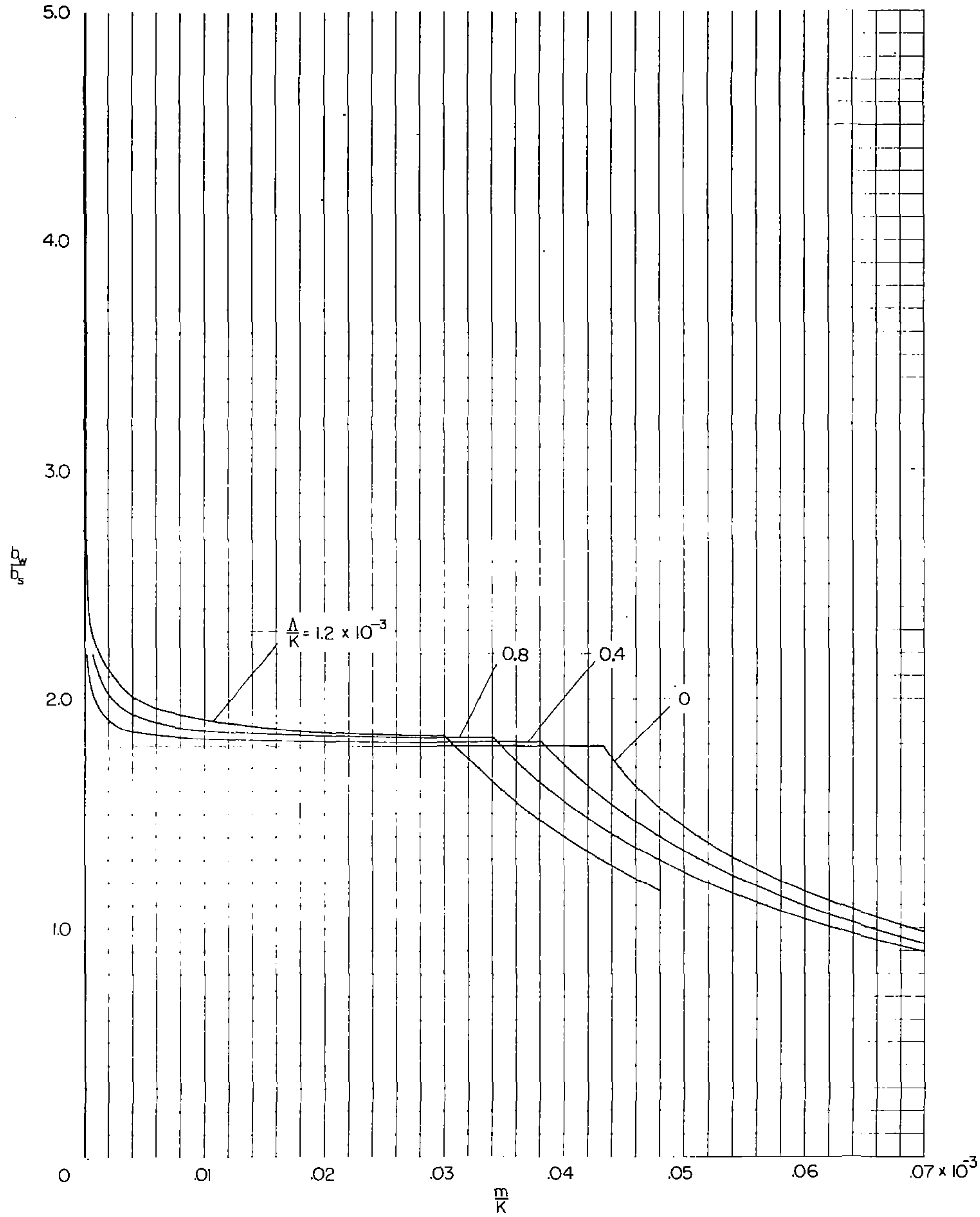
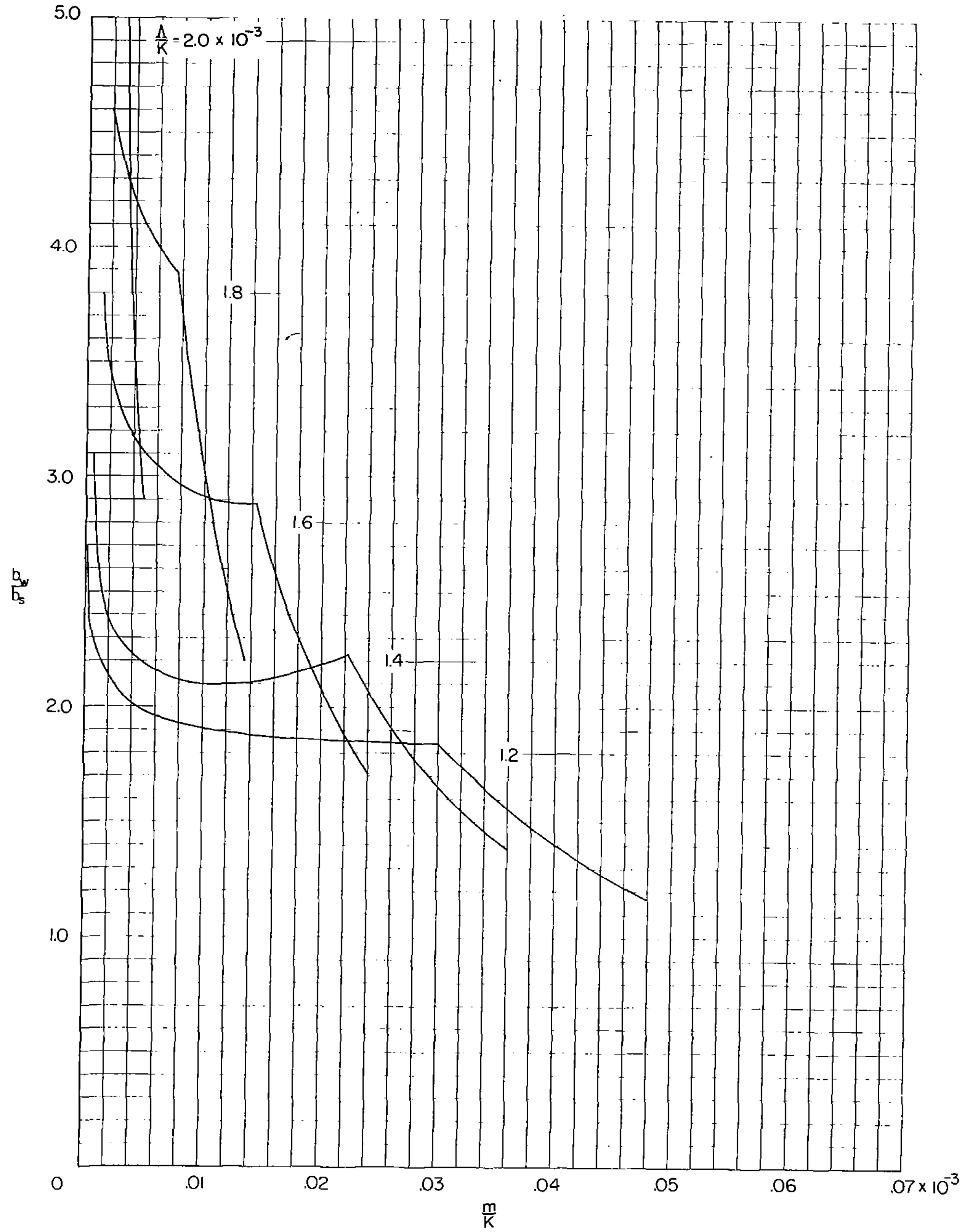


Figure 3.- Solidity as a function of dimensionless structural index and temperature parameter for optimum beams obtained from second analysis. Dashed lines separate regions; circular points indicate beams used in illustrative examples.



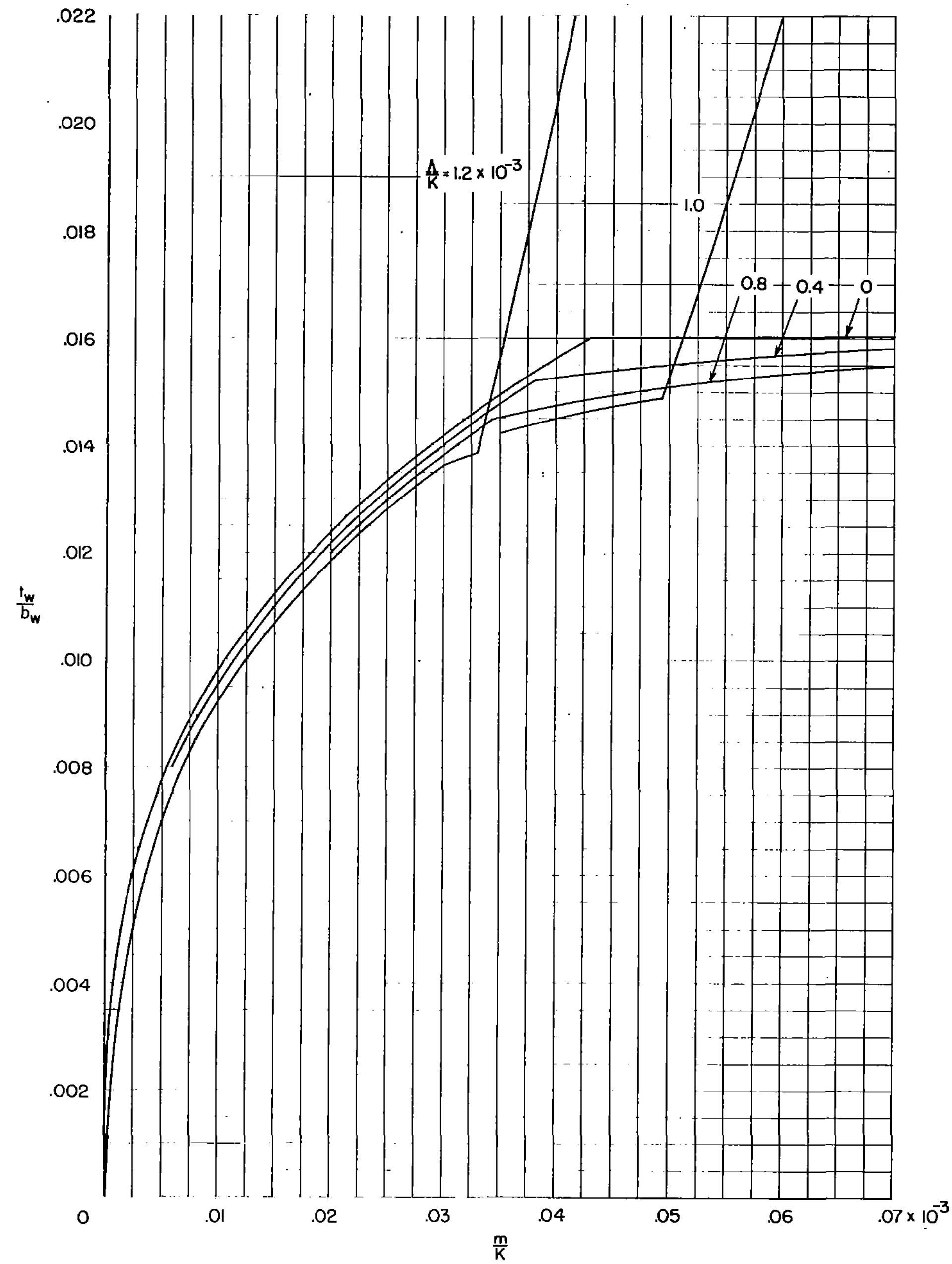
(a) $0 \leqq \frac{\Lambda}{K} \leqq 0.0012.$

Figure 4.- Ratio of beam depth to web spacing as a function of dimensionless structural index for optimum beams obtained from second analysis.



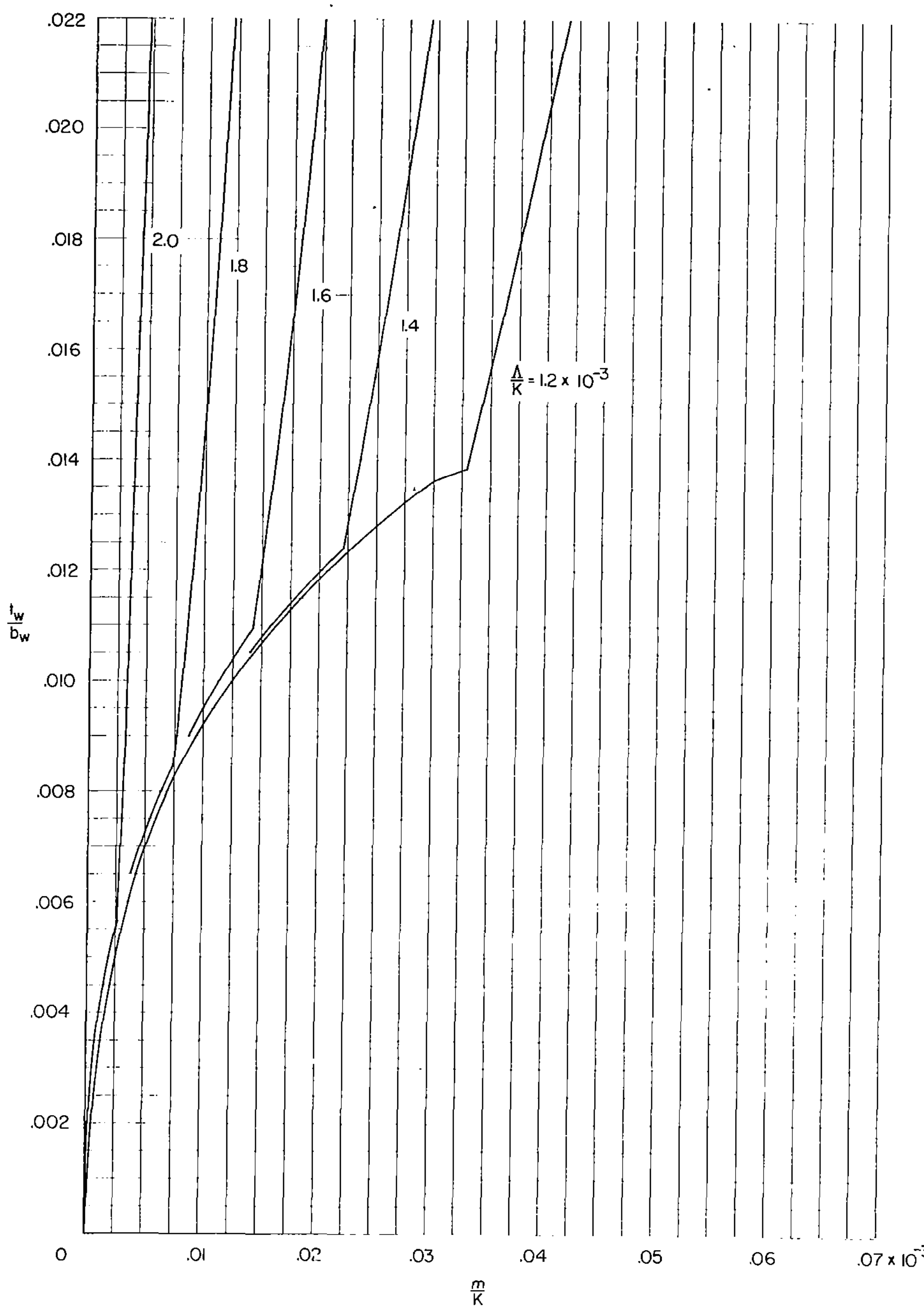
(b) $0.0012 \leqq \frac{\Lambda}{K} \leqq 0.0020.$

Figure 4.- Concluded.



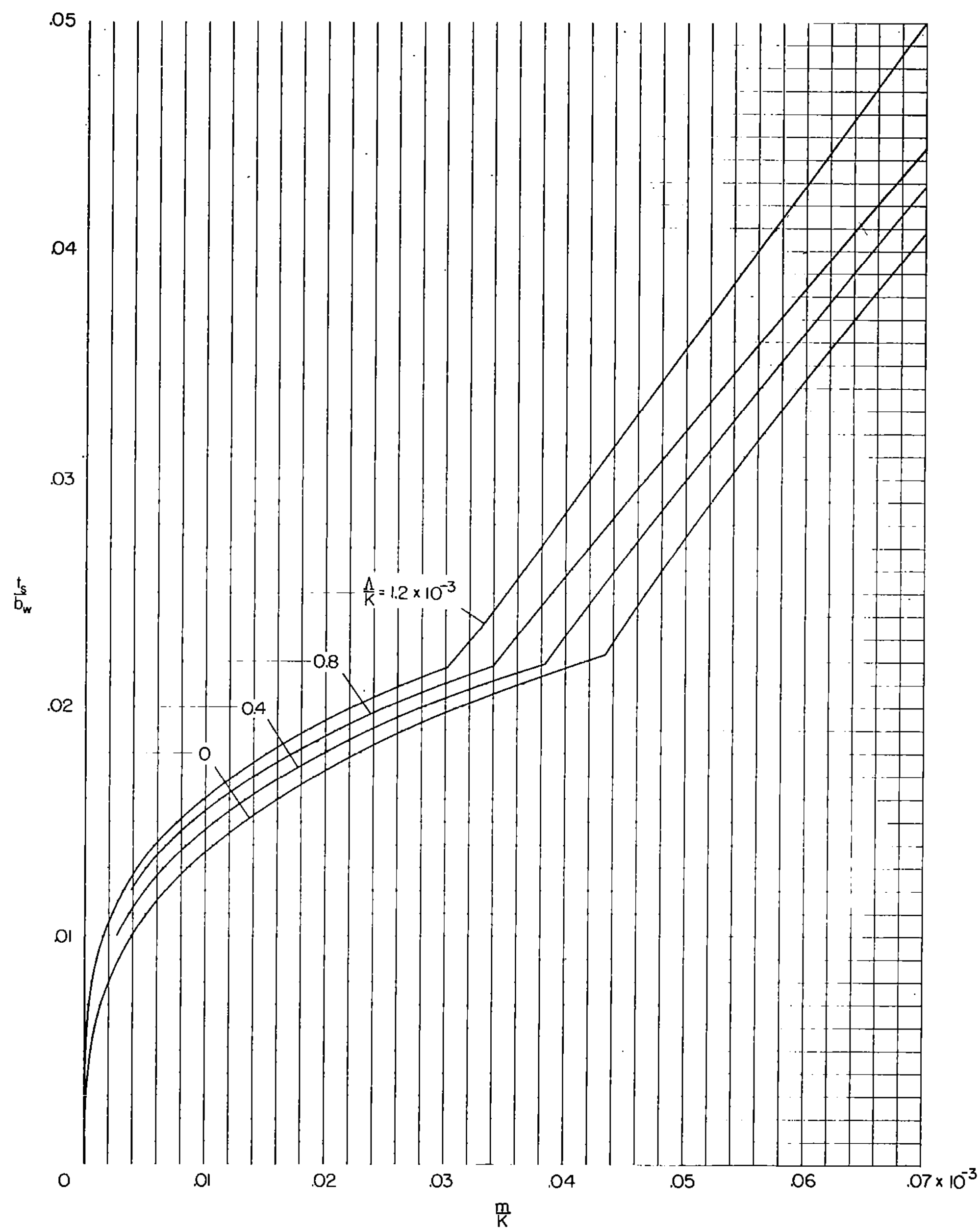
(a) $0 \leqq \frac{\Lambda}{K} \leqq 0.0012.$

Figure 5.- Ratio of web thickness to beam depth as a function of dimensionless structural index for optimum beams obtained from second analysis.



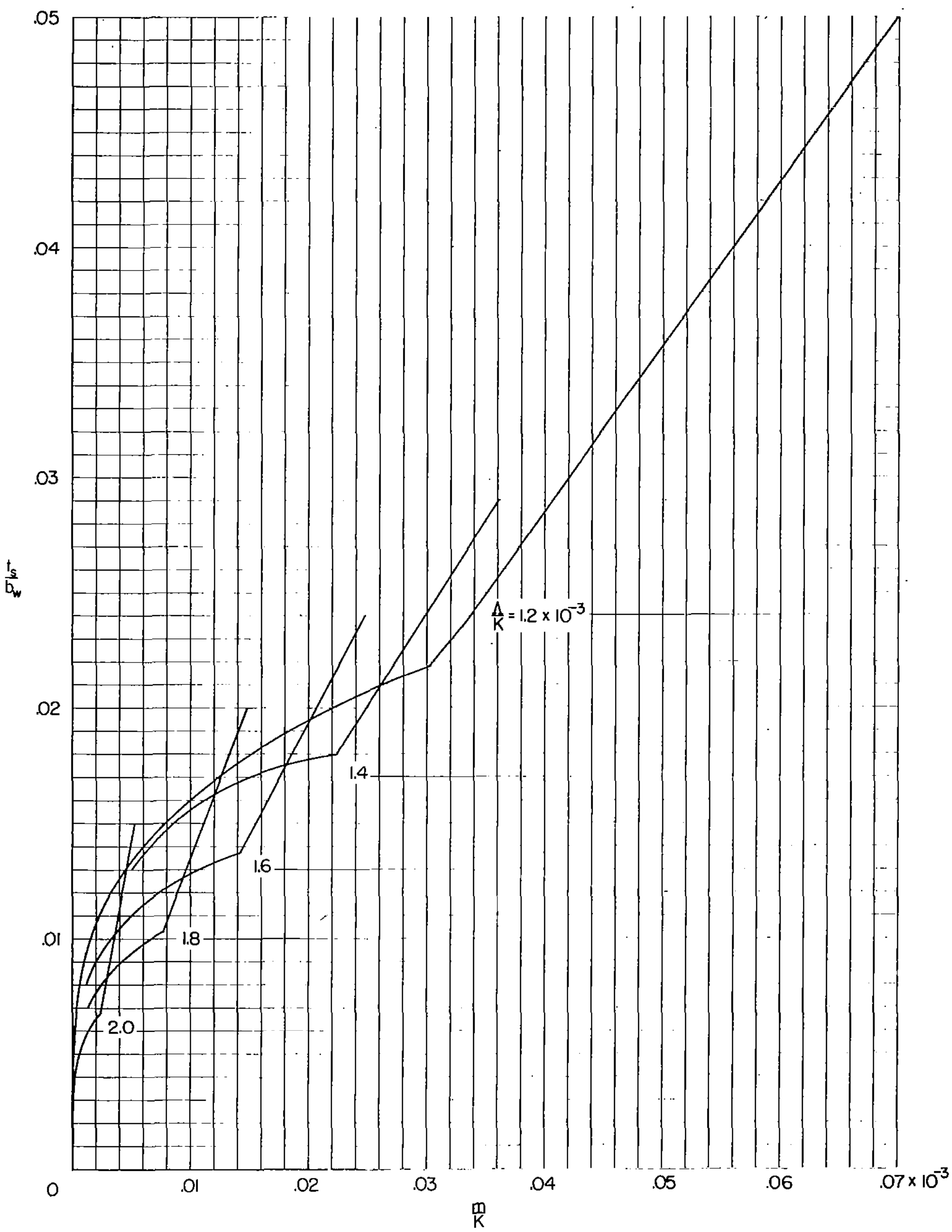
(b) $0.0012 \leqq \frac{\Lambda}{K} \leqq 0.0020.$

Figure 5.- Concluded.



(a) $0 \leqq \frac{\Lambda}{K} \leqq 0.0012$.

Figure 6.- Ratio of skin thickness to beam depth as a function of dimensionless structural index for optimum beams obtained from second analysis.



(b) $0.0012 \leqq \frac{\Lambda}{K} \leqq 0.0020$.

Figure 6.- Concluded.

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